A NOTE ON THE ULTRAHYPERBOLIC WAVE EQUATION IN $3+3$ DIMENSIONS

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This note concerns the following two operators on $S^3 \times S^3$ recently studied in connection with twistor theory in [Sparling(2007)], [Sparling(2006)]. For $x, y \in S^3$ and $f \in C^\infty(S^3 \times S^3)$, define

$$Tf(x,y) = \int_{S^3} f(xg, gy)\, dg \tag{1}$$

$$\Box f = \Delta_x f - \Delta_y f \tag{2}$$

where dg is the invariant probability measure on the Lie group $S^3 \cong SU(2)$ and Δ is the Laplace-Beltrami operator on S^3. Our main theorem is the following

Theorem 1. *Acting on smooth functions,* $\ker T = \operatorname{im} \Box$ *and* $\ker \Box = \operatorname{im} T$*; that is,* $T, \Box$ *form an exact couple.*

The strategy is to first prove that the theorem holds for T and $\Box$ restricted to the subspace $\mathscr{H}$ consisting of finite linear combinations of spherical harmonics, and then to extend this first, by a density argument, to the Sobolev spaces $H^s = H^s(S^3 \times S^3)$. The result then follows from the Sobolev lemma $C^\infty = \bigcap_s H^s$. Both operators are understood as acting on H^s in a weak sense.

We first review some of the theory of spherical harmonics in order to fix notation. Let $\mathscr{P}_k$ be the space of all polynomials f on $\mathbb{R}^4$ homogeneous of degree k: $f(tx) = t^k f(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^4$. Let $\mathscr{H}_k \subset \mathscr{P}_k$ be the subset consisting of all harmonic polynomials: $\Delta_{\mathbb{R}^4} f = 0$, where $\Delta_{\mathbb{R}^4}$ is the ordinary Euclidean Laplacian. The following theorem summarizes the basic properties of spherical harmonics:

Proposition 1.

(a) *Every polynomial* $p \in \mathscr{P}_k$ *can be written uniquely as*

$$p(x) = p_0(x) + |x|^2 p_1(x) + \cdots + |x|^{\lfloor k/2 \rfloor} p_{\lfloor k/2 \rfloor}$$

where $p_j \in \mathscr{H}_{k-2j}$.

(b) *If* $p \in \mathscr{H}_k$*, then* $\Delta_{S^3} p|_{S^3} = -k(k+2)p|_{S^3}$*, where* Δ_{S^3} *is the Laplace-Beltrami operator on the unit sphere in* $\mathbb{R}^4$.

By the Stone-Weierstrass theorem, spherical harmonics are dense in $C(S^3)$, and hence also in $L^2(S^3) = H^0(S^3)$. The different eigenspaces of the Laplace-Beltrami operator are orthonormal in $L^2(S^3)$, and so the spaces $\mathscr{H}_k$ give an orthogonal decomposition

$$L^2(S^3) = \widehat{\bigoplus_{k=0}^{\infty}} \mathscr{H}_k$$

in terms of which it is possible to decompose uniquely any $f \in L^2(S^3)$ into the Fourier series

$$f = \sum_{k=0}^{\infty} a_k Y_k \tag{3}$$

where $Y_k \in \mathscr{H}_k$ are normalized to have unit norm. In particular, the spaces $\mathscr{H}_k$ exhaust the eigenspaces of Δ_{S^3}. The Sobolev spaces admit the following characterization in terms of the Fourier series:

Lemma 1. *Let $s \geq 0$ and $f \in H^0(S^3)$ have Fourier decomposition* (3)*. Then $f \in H^s(S^3)$ if and only if $\sum_{k=0}^{\infty} |a_k|^2 k^{2s} < \infty$.*

Now we turn attention to $S^3 \times S^3$. The eigenvalues of $\Delta_{S^3 \times S^3} = \Delta_x + \Delta_y$ are precisely the tensor products $\mathscr{H}_k \otimes \mathscr{H}_\ell$. Thus to each $f \in H^s(S^3 \times S^3)$, there is a double series expansion in spherical harmonics

$$f = \sum_{k,\ell=0}^{\infty} a_{k\ell} Y_k \otimes Y_\ell$$

where the Y_k are normalized elements of $\mathscr{H}_k$. As a result, there is an isomorphism of Hilbert spaces

$$H^s(S^3 \times S^3) \cong H^s(S^3) \,\widehat{\otimes}\, H^s(S^3),$$

where the tensor product appearing on the right is the completion of the usual tensor product under the Hilbert inner product defined on simple tensors by $\langle a \otimes b, c \otimes d \rangle_s = \langle a, c\rangle_s \langle b, d\rangle_s$.

Lemma 2.

(a) The operator $T : H^s(S^3 \times S^3) \to H^s(S^3 \times S^3)$ is a bounded self-adjoint operator.

(b) The kernel of T contains the subspace $\widehat{\bigoplus}_{k \neq \ell} \mathscr{H}_k \otimes \mathscr{H}_\ell$, where the closure is taken in the H^s norm.

Proof. (a) It is sufficient to prove boundedness and self-adjointness in L^2, since the proof for H^s follows by applying the L^2 result to $(1 - \Delta_x - \Delta_y)^{s/2} f$. By the integral Minkowski inequality,

$$\begin{aligned} \|Tf\|_2 &= \left\{ \int_{S^3 \times S^3} \left| \int_{S^3} f(xg, gy)\, dg \right|^2 dxdy \right\}^{1/2} \\ &\leq \int_{S^3} \left\{ \int_{S^3 \times S^3} |f(xg, gy)|^2\, dxdy \right\}^{1/2} dg = \|f\|_2. \end{aligned}$$

This proves boundedness. Self-adjointness follows from an application of Fubini's theorem.

(b) Suppose first that $f(x,y) = u(x)v(y)$ where $u \in \mathscr{H}_k$, $v \in \mathscr{H}_\ell$, and $k \neq \ell$. Since Δ_{S^3} is invariant under both left and right translation in S^3, for fixed $x, y \in S^3$ we have $R_x u \in \mathscr{H}_k$ and $L_y v \in \mathscr{H}_\ell$ so that by orthogonality of the $\mathscr{H}_k$,

$$Tf(x,y) = \int_{S^3} R_x u L_y v = 0.$$

The result now follows from the boundedness of T.

□

A sharper result, completely classifying the range of T as well, requires knowing that the range is closed. Let $E_{k,\ell} : H^s \to \mathscr{H}_k \otimes \mathscr{H}_\ell$ be the orthogonal projection. The following gives the operator T as a Fourier multiplier:

Theorem 2. *For any s, $T \in \mathscr{B}(H^s, H^{s+1})$. The kernel of T is $\widehat{\bigoplus}_{k\neq\ell}\mathscr{H}_k \otimes \mathscr{H}_\ell$ and the image is $\widehat{\bigoplus}_k \mathscr{H}_k \otimes \mathscr{H}_k$. In fact,*

$$T = \sum_{k=0}^{\infty} \frac{1}{k+1} R_k E_{k,k}$$

where each R_k is a reflection operator (self-adjoint involution) in the space $\mathscr{H}_k \otimes \mathscr{H}_k$.

Assume for the moment that the image of $T \in \mathscr{B}(H^{s-1}, H^s)$ contains the space $B_s = \widehat{\bigoplus}_k \mathscr{H}_k \otimes \mathscr{H}_k$. By Lemma 2, the kernel of T in H^s contains the subspace $A_s = \widehat{\bigoplus}_{k\neq\ell}\mathscr{H}_k \otimes \mathscr{H}_\ell$. To prove the opposite inclusion, note that the spaces A_s and B_s are orthogonal complements. So any $x \in \ker T$ has the form $x = a \oplus b$ for $b \in B_s$. By assumption, $b = Tc$ for some $c \in H^{s-1}$. Then applying T gives $0 = Tx = T^2 c$, and this implies that $c = 0$ because T is self-adjoint. Therefore $x \in A_s$, and so $\ker T = A_s$, as claimed. Effectively the same argument shows that, under the same assumption, the image of T on H^{s-1} must then be *equal* to B_s. It is therefore enough to show that the image of T in H^s contains the subspace B_s.

The proof of this fact along with last assertion of the theorem employs the zonal spherical harmonics $Z_x^{(k)}(y)$ on S^3, defined for $f \in C^\infty(S^3)$ by

$$\int_{S^3} Z_x^{(k)}(y) f(y)\, dy = (E_k f)(y)$$

where E_k is the orthogonal projection onto $\mathscr{H}_k$. They satisfy the following properties

Lemma 3 ([Stein and Weiss(1971)])**.**

(a) For each fixed x, $y \mapsto Z_x^{(k)}(y)$ is in $\mathscr{H}_k$.
(b) For each rotation $\rho \in SO(4)$, $Z_{\rho x}^{(k)}(\rho y) = Z_x^{(k)}(y)$.
(c) Conversely, any function satisfying these two properties is a constant multiple of $Z_x^{(k)}(y)$.
(d) The following integral identities hold:

$$(k+1)^2 = \int_{S^3} |Z_y^{(k)}(x)|^2\, dx = \int_{S^3} Z_x^{(k)}(x)\, dx = Z_e^{(k)}(e).$$

The zonal harmonics enter the proof by integrating Tf, for $f \in \mathscr{H}_k \otimes \mathscr{H}_k$ against the product of zonal harmonics $Z_{x'}^{(k)}(x) Z_{y'}^{(k)}(y)$. Fubini's theorem, followed by an

obvious change of variables gives

$$\begin{aligned}
\iint_{S^3\times S^3} Z^{(k)}_{x'}(x) Z^{(k)}_{y'}(y) Tf(x,y)\,dxdy &= \int_{S^3}\iint_{S^3\times S^3} Z^{(k)}_{x'}(x) Z^{(k)}_{y'}(y) f(xg,gy)\,dxdydg \\
&= \int_{S^3}\iint_{S^3\times S^3} Z^{(k)}_{x'}(xg^{-1}) Z^{(k)}_{y'}(g^{-1}y) f(x,y)\,dxdydg \\
&= \iint_{S^3\times S^3} f(x,y)\,dxdy \int_{S^3} Z^{(k)}_{x'g}(x) Z^{(k)}_{gy'}(y) dg \\
&\qquad \text{by invariance of } Z^{(k)} \\
&= \iint_{S^3\times S^3} f(x,y)\,dxdy \int_{S^3} Z^{(k)}_{g}(\bar{x}'x) Z^{(k)}_{gy'}(gy) dg \\
&= \iint_{S^3\times S^3} f(x,y)\,dxdy \int_{S^3} Z^{(k)}_{\bar{x}'xy'}(y) dg
\end{aligned}$$

by the reproducing property of $Z^{(k)}$. The bar denotes quaternionic conjugation, which is the same as inversion in the group S^3. Thus for $f \in \mathscr{H}_k \otimes \mathscr{H}_k$, Tf is obtained by integrating against the kernel

$$Z^{(k)}_{\bar{x}'xy'}(y) = Z^{(k)}_{\bar{x}'x}(y\bar{y}') = Z^{(k)}_{xy'}(x'y).$$

To establish the theorem, it is sufficient to show that $T^2|_{\mathscr{H}_k\otimes\mathscr{H}_k} = \lambda_k^2 \operatorname{Id}_{\mathscr{H}_k\otimes\mathscr{H}_k}$. Applying T twice is the same as integrating against the kernel in the variables (x'', y'') defined by

$$K(x'', y''; x, y) = \iint_{S^3\times S^3} Z^{(k)}_{x'y''}(x''y') Z^{(k)}_{xy'}(x'y)\,dx'dy'.$$

To show that $T^2|_{\mathscr{H}_k\otimes\mathscr{H}_k} = \lambda_k^2 \operatorname{Id}$, it is sufficient to prove that $K(x'', y''; x, y) = \lambda_k^2 Z_x(x'') Z_y(y'')$. By the characterization in Lemma 3(c) it is then enough to show that for all $g \in S^3$,

$$\begin{aligned}
K(x'', y''; x, y) &= K(x''g, y''; xg, y) = K(gx'', y''; gx, y) \\
&= K(x'', y''g; x, yg) = K(x'', gy''; g, gy),
\end{aligned}$$

since every special orthogonal transformation has the form $x \mapsto gxh$ on S^3 for some g, h. Clearly, by symmetry in the variables, it is enough to show the first two identities. By invariance of the Haar measure,

$$\begin{aligned}
K(x''g, y''; xg, y) &= \iint_{S^3\times S^3} Z^{(k)}_{x'y''}(x''gy') Z^{(k)}_{xgy'}(x'y)\,dx'dy' \\
&= \iint_{S^3\times S^3} Z^{(k)}_{x'y''}(x''\tilde{y}') Z^{(k)}_{x\tilde{y}'}(x'y)\,dx'd\tilde{y}' = K(x'', y''; x, y)
\end{aligned}$$

and,

$$\begin{aligned}
K(gx'', y''; gx, y) &= \iint_{S^3\times S^3} Z^{(k)}_{x'y''}(gx''y') Z^{(k)}_{gxy'}(x'y)\,dx'dy' \\
&= \iint_{S^3\times S^3} Z^{(k)}_{g^{-1}x'y''}(x''y') Z^{(k)}_{xy'}(g^{-1}x'y)\,dx'dy' \\
&= \iint_{S^3\times S^3} Z^{(k)}_{\tilde{x}'y''}(x''y') Z^{(k)}_{xy'}(\tilde{x}'y)\,d\tilde{x}'dy' = K(x'', y''; x, y).
\end{aligned}$$

So on $\mathscr{H}_k \otimes \mathscr{H}_k$, $T^2 = \lambda_k^2 \operatorname{Id}$.

It remains to calculate λ_k. For this, Lemma 3(d) gives

$$\begin{aligned}
(k+1)^4\lambda_k^2 &= \iint_{S^3\times S^3} K(x,y;x,y)\,dxdy \\
&= \iiiint_{(S^3)^4} Z^{(k)}_{x'y}(xy')Z^{(k)}_{xy'}(x'y)\,dx'dy'dxdy \\
&= \iiiint_{(S^3)^4} Z^{(k)}_{x'}(xy'\bar{y})Z^{(k)}_{xy'}(x'y)\,dx'dy'dxdy \\
&= \iiint_{(S^3)^3} Z^{(k)}_{xy'}(xy')\,dy'dxdy \\
&= (k+1)^2
\end{aligned}$$

so that $\lambda_k = (k+1)^{-1}$, as required.

References

[Sparling(2007)] G. A. J. Sparling, "Germ of a synthesis: space-time is spinorial, extra dimensions are time-like", *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **463** (2007), no. 2083, 1665–1679.

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